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Linear Asymptotic Equilibrium and Uniform, Exponential, and Strict Stability of Linear Difference Systems

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Abstract—Following an idea originated by Conti for continuous matrix functions, an equivalence relation called *summable similarity* is defined on pairs of $n \times n$ matrix functions A and B. Special cases of the results show that if A and B are summably similar and the system (A) $\Delta x_m = A_m x_m$, $m = 0, 1, 2, \dots$ is uniformly, exponentially, or strictly stable, or has linear asymptotic equilibrium, then the system (B) $\Delta y_m = B_m y_m$, $m = 0, 1, 2, \dots$ has the same property. More generally, the same conclusion is obtained under weaker conditions relating A and B.

Keywords—Difference equations, Uniform stability, Exponential stability, Strict stability, Linear asymptotic equilibrium.

1. INTRODUCTION

Conti [1] defined two $n \times n$ matrix functions A and B on $[0, \infty)$ to be t_∞-*similar* if there is an $n \times n$ matrix function S defined on $[0, \infty)$ such that S' is continuous, S and S^{-1} are bounded on $[0, \infty)$, and

$$\int_0^\infty \|S' + SB - AS\| \, dt < \infty. \tag{1}$$

Conti [2] showed that t_∞-similarity is an equivalence relation that preserves strict and uniform stability of a linear homogeneous system $x' = A(t)x$; that is, if the system $x' = A(t)x$ has either of these properties and B is t_∞-similar to A, then so does the system $y' = B(t)y$. It can also be shown that t_∞-similarity preserves exponential stability. The author [3] weakened (1) and introduced a definition called t_∞-*quasisimilarity* that is not symmetric or transitive, but still preserves these properties. In this paper we give analogs of some of the results in [1,3] for systems of difference equations.

2. PRELIMINARY RESULTS

Throughout this paper $\mathcal{N}$ denotes the nonnegative integers, $\mathcal{M}^{n\times n}$ is the set of $n \times n$ matrix functions (with real or complex entries) defined on $\mathcal{N}$,

$$\mathcal{S} = \left\{ S \in \mathcal{M}^{n\times n} : S \text{ and } S^{-1} \text{ are bounded} \right\},$$

$$\mathcal{I} = \left\{ F \in \mathcal{M}^{n\times n} : \sum_{m=0}^{\infty} F_m \text{ exists} \right\}, \quad \text{and} \quad \mathcal{A} = \left\{ F \in \mathcal{M}^{n\times n} : \sum_{m=0}^{\infty} \|F_m\| < \infty \right\}.$$

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We are interested in relating stability properties of two $n \times n$ systems of difference equations

$$\Delta x_m = A_m x_m, \qquad m \geq 0, \qquad \text{and} \tag{A}$$
$$\Delta y_m = B_m y_m, \qquad m \geq 0, \tag{B}$$

where Δ is the forward difference operator. We assume that $I + A_m$ and $I + B_m$ are invertible for every $m \geq 0$. This guarantees that $X, Y \in \mathcal{M}^{n\times n}$ defined by

$$X_0 = I, \qquad X_m = (I + A_{m-1}) \dots (I + A_0), \qquad m \geq 1,$$
$$Y_0 = I, \qquad Y_m = (I + B_{m-1}) \dots (I + B_0), \qquad m \geq 1,$$

are fundamental matrices for (A) and (B), respectively. If m_0 is a fixed nonnegative integer, then the solutions of (A) and (B) satisfy

$$x_m = X_m X_{m_0}^{-1} x_{m_0} \quad \text{and} \quad y_m = Y_m Y_{m_0}^{-1} y_{m_0}, \qquad m \geq 0,$$

respectively.

The properties that we consider are defined as follows.

DEFINITION 1.

(a) *(A) is uniformly stable if there is a constant C_A independent of m_0 such that every solution of (A) satisfies $\|x_m\| \leq C_A \|x_{m_0}\|$, $m \geq m_0 \geq 0$.*

(b) *(A) is exponentially stable if there are constants C_A and ρ independent of m_0, with $0 < \rho < 1$, such that every solution of (A) satisfies $\|x_m\| \leq C_A \|x_{m_0}\| \rho^{m_0 - m}$, $m \geq m_0$.*

(c) *(A) is strictly stable if there is a constant C_A such that every solution of (A) satisfies $\|x_m\| \leq C_A \|x_{m_0}\|$, $m, m_0 \geq 0$.*

(d) *(A) has linear asymptotic equilibrium if every nontrivial solution of (A) approaches a nonzero limit as $m \to \infty$.*

We omit the elementary proof of the following theorem.

THEOREM 1.

(a) *(A) is uniformly stable if and only if there is a constant $C_A > 0$ such that $\|X_j X_i^{-1}\| \leq C_A$, $0 \leq i \leq j$.*

(b) *(A) is exponentially stable if and only if there are constants C_A and ρ such that $C_A > 0$, $0 < \rho < 1$ and $\|X_j X_i^{-1}\| \leq C_A \rho^{i-j}$, $0 \leq i \leq j$.*

(c) *(A) is strictly stable if and only if there is a constant $C_A > 0$ such that $\|X_j X_i^{-1}\| \leq C_A$, $i, j \geq 0$, or, equivalently, if and only if X and X^{-1} are both bounded on $\mathcal{N}$.*

(d) *(A) has linear asymptotic equilibrium if and only if $\lim_{j\to\infty} X_j$ exists and is invertible.*

The following definition is a discrete analog of Conti's definition of t_∞-similarity.

DEFINITION 2. *If $A, B \in \mathcal{M}^{n\times n}$ then B is summably similar to A (written as $B \sim A$) if there is an $S \in \mathcal{S}$ such that $F \in \mathcal{M}^{n\times n}$ defined by*

$$F_m = \Delta S_m + S_{m+1} B_m - A_m S_m \tag{2}$$

is in $\mathcal{A}$.

THEOREM 2. *Summable similarity is an equivalence relation.*

PROOF. Taking $B = A$ and $S_m = I$ in (2) shows that $A \sim A$. To show that $B \sim A$ implies that $A \sim B$, suppose that (2) holds for some $S \in \mathcal{S}$. Then,

$$\begin{aligned} -S_{m+1}^{-1} F_m S_m^{-1} &= -S_{m+1}^{-1} (\Delta S_m) S_m^{-1} - B_m S_m^{-1} + S_{m+1}^{-1} A_m \\ &= \Delta S_m^{-1} + S_{m+1}^{-1} A_m - B_m S_m^{-1}. \end{aligned}$$

Since $S^{-1} \in \mathcal{S}$ and $F \in \mathcal{A}$, it follows that $\sum_{m=0}^{\infty} \|S_{m+1}^{-1} F_m S_m^{-1}\| < \infty$. Therefore, $A \sim B$.

Now suppose that $B \sim A$ and $C \sim B$. Let S and T be in $\mathcal{S}$ such that the matrix functions F and G defined by

$$F_m = \Delta S_m + S_{m+1}B_m - A_m S_m,$$
$$G_m = \Delta T_m + T_{m+1}C_m - B_m T_m$$

are in $\mathcal{A}$. Then

$$F_m T_m = (\Delta S_m)T_m + S_{m+1}B_m T_m - A_m S_m T_m,$$
$$S_{m+1}G_m = S_{m+1}(\Delta T_m) + S_{m+1}T_{m+1}C_m - S_{m+1}B_m T_m,$$

and

$$\begin{aligned} S_{m+1}G_m + F_m T_m &= S_{m+1}(\Delta T_m) + (\Delta S_m)T_m + S_{m+1}T_{m+1}C_m - A_m S_m T_m \\ &= \Delta(S_m T_m) + S_{m+1}T_{m+1}C_m - A_m S_m T_m. \end{aligned}$$

Since $S, T \in \mathcal{S}$ and $F, G \in \mathcal{A}$, it follows that $\sum_{m=0}^{\infty} \|S_{m+1}G_m + F_m T_m\| < \infty$. Since $ST \in \mathcal{N}$, this implies that $C \sim A$. ∎

We will see that summable similarity preserves linear asymptotic equilibrium and uniform, exponential, and strict stability. However, these results do not require the relationship of B to A to be symmetric or transitive. We make the following weaker assumption.

ASSUMPTION 1. *There is an $S \in \mathcal{S}$ such that*

$$F_m^{(0)} = \Delta S_m + S_{m+1}B_m - A_m S_m \tag{3}$$

defines an element of $\mathcal{I}$. Either $F^{(0)} \in \mathcal{A}$ (so $B \sim A$, and we define $p = 0$), or there is a positive integer p such that the $n \times n$ matrix functions $F^{(1)}, \ldots, F^{(p)}$ defined by

$$Q_m^{(r)} = \sum_{k=m}^{\infty} F_k^{(r-1)} \qquad \text{and}$$
$$F_m^{(r)} = Q_{m+1}^{(r)} B_m - A_m Q_m^{(r)} \tag{4}$$

are in $\mathcal{I}$, and $F^{(p)} \in \mathcal{A}$.

LEMMA 1. *Suppose that Assumption 1 holds. Define*

$$\Gamma^{(0)} = I \quad \text{and} \quad \Gamma^{(r)} = I + S^{-1} \sum_{l=1}^{r} Q^{(l)}, \qquad 1 \le r \le p.$$

Then

$$\Gamma_j^{(p)} Y_j = S_j^{-1} X_j \left[X_i^{-1} S_i \Gamma_i^{(p)} Y_i + \sum_{m=i}^{j-1} X_{m+1}^{-1} F_m^{(p)} Y_m \right], \qquad 0 \le i \le j. \tag{5}$$

PROOF. Since $A_m = (\Delta X_m)X_m^{-1}$ and $B_m = (\Delta Y_m)Y_m^{-1}$, we can rewrite (3) as

$$F_m^{(0)} = \Delta S_m + S_{m+1}(\Delta Y_m)Y_m^{-1} - (\Delta X_m)X_m^{-1} S_m.$$

Therefore

$$\begin{aligned} X_{m+1}^{-1} F_m^{(0)} Y_m &= X_{m+1}^{-1}(\Delta S_m)Y_m + X_{m+1}^{-1} S_{m+1}(\Delta Y_m) - X_{m+1}^{-1}(\Delta X_m)X_m^{-1} S_m Y_m \\ &= X_{m+1}^{-1}\Delta(S_m Y_m) + \left(\Delta X_m^{-1}\right) S_m Y_m = \Delta\left(X_m^{-1} S_m Y_m\right); \end{aligned}$$

that is,

$$\Delta\left(X_m^{-1}S_mY_m\right) = X_{m+1}^{-1}F_m^{(0)}Y_m. \tag{6}$$

Summing this from $m = i$ to $m = j-1$ yields

$$X_j^{-1}S_jY_j = X_i^{-1}S_iY_i + \sum_{m=i}^{j-1} X_{m+1}^{-1}F_m^{(0)}Y_m.$$

Therefore

$$Y_j = S_j^{-1}X_j\left[X_i^{-1}S_iY_i + \sum_{m=i}^{j-1} X_{m+1}^{-1}F_m^{(0)}Y_m\right].$$

This completes the proof if $p = 0$.

Now, suppose that $p \geq 1$ and we have established (5) with p replaced by $p-1$; that is,

$$\Gamma_j^{(p-1)}Y_j = S_j^{-1}X_j\left[X_i^{-1}S_i\Gamma_i^{(p-1)}Y_i + \sum_{m=i}^{j-1} X_{m+1}^{-1}F_m^{(p-1)}Y_m\right]. \tag{7}$$

Note, that

$$\begin{aligned}
\Delta\left(X_m^{-1}Q_m^{(p)}Y_m\right) &= X_{m+1}^{-1}\Delta\left(Q_m^{(p)}Y_m\right) + \left(\Delta X_m^{-1}\right)Q_m^{(p)}Y_m \\
&= X_{m+1}^{-1}Q_{m+1}^{(p)}\Delta Y_m + X_{m+1}^{-1}\left(\Delta Q_m^{(p)}\right)Y_m - X_{m+1}^{-1}(\Delta X_m)X_m^{-1}Q_m^{(p)}Y_m \\
&= X_{m+1}^{-1}\left(Q_{m+1}^{(p)}B_m - F_m^{(p-1)} - A_mQ_m^{(p)}\right)Y_m \\
&= X_{m+1}^{-1}\left(F_m^{(p)} - F_m^{(p-1)}\right)Y_m,
\end{aligned}$$

so

$$X_{m+1}^{-1}F_m^{(p-1)}Y_m = X_{m+1}^{-1}F_m^{(p)}Y_m - \Delta\left(X_m^{-1}Q_m^{(p)}Y_m\right).$$

Therefore

$$\sum_{m=i}^{j-1} X_{m+1}^{-1}F_m^{(p-1)}Y_m = -X_j^{-1}Q_j^{(p)}Y_j + X_i^{-1}Q_i^{(p)}Y_i + \sum_{m=i}^{j-1} X_{m+1}^{-1}F_m^{(p)}Y_m.$$

Substituting this into (7) and collecting terms involving Y_j on the left yields (5). ∎

3. MAIN RESULTS

THEOREM 3. *If Assumption 1 holds and (A) is uniformly or exponentially stable then (B) is also.*

PROOF. From Theorem 1 and our hypothesis on (A) there are positive constants $C_A > 0$ and ρ such that

$$\left\|X_jX_i^{-1}\right\| \leq C_A\rho^{i-j}, \qquad 0 \leq i \leq j. \tag{8}$$

(If (A) is exponentially stable then $\rho < 1$; if (A) is uniformly stable but not exponentially stable then $\rho = 1$.) Since $\lim_{m\to\infty} Q_m^{(r)} = 0$ $(1 \leq r \leq p)$ and S^{-1} is bounded, it follows that $(\Gamma^{(p)})_i^{-1}$ exists and is bounded for i sufficiently large, say $i \geq i_0 \geq 0$. We will show that there is a constant C_B such that

$$\left\|Y_jY_i^{-1}\right\| \leq C_B\rho^{i-j}, \qquad i_0 \leq i \leq j. \tag{9}$$

Then Theorem 1 will imply the conclusion.

From (5)

$$Y_jY_i^{-1} = \left(\Gamma^{(p)}\right)_j^{-1}\left(S_j^{-1}X_jX_i^{-1}S_i\Gamma_i^{(p)} + \sum_{m=i}^{j-1} S_j^{-1}X_jX_{m+1}^{-1}F_m^{(p)}Y_mY_i^{-1}\right)$$

for $i_0 \le i \le j$. Since $S, S^{-1}, \Gamma^{(p)}$, and $(\Gamma^{(p)})^{-1}$ are bounded, this implies that there are constants α and β such that

$$\left\|Y_jY_i^{-1}\right\| \le \frac{\alpha}{C_A}\left\|X_jX_i^{-1}\right\| + \frac{\beta}{C_A}\sum_{m=i}^{j-1}\left\|X_jX_{m+1}^{-1}\right\|\left\|F_m^{(p)}\right\|\left\|Y_mY_i^{-1}\right\|, \qquad i_0 \le i \le j.$$

From this and (8),

$$\left\|Y_jY_i^{-1}\right\| \le \alpha\rho^{i-j} + \beta\sum_{m=i}^{j-1}\rho^{m+1-j}\left\|F_m^{(p)}\right\|\left\|Y_mY_i^{-1}\right\|, \qquad i_0 \le i \le j,$$

which we rewrite as

$$\rho^{j-i}\left\|Y_jY_i^{-1}\right\| \le u_{ij}, \qquad i_0 \le i \le j, \tag{10}$$

with

$$u_{ij} = \alpha + \beta\rho\sum_{m=i}^{j-1}\left\|F_m^{(p)}\right\|\rho^{m-i}\left\|Y_mY_i^{-1}\right\|. \tag{11}$$

Then (10) and (11) imply that

$$u_{i,j+1} - u_{i,j} = \beta\rho\left\|F_j^{(p)}\right\|\rho^{j-i}\left\|Y_jY_i^{-1}\right\| \le \beta\rho\left\|F_j^{(p)}\right\|u_{ij}, \qquad i_0 \le i \le j.$$

Therefore

$$u_{i,j+1} \le \left(1 + \beta\rho\left\|F_j^{(p)}\right\|\right)u_{ij} \le u_{ij}e^{\beta\rho\left\|F_j^{(p)}\right\|}, \qquad i_0 \le i \le j.$$

Since $u_{ii} = \alpha$ this implies that

$$u_{ij} \le \alpha\prod_{m=i}^{j-1} e^{\beta\rho\left\|F_m^{(p)}\right\|}, \qquad i_0 \le i \le j. \tag{12}$$

Since $F^{(p)} \in \mathcal{A}$, (10) and (12) imply (9) with

$$C_B = \alpha\exp\left(\beta\rho\sum_{m=0}^{\infty}\left\|F_m^{(p)}\right\|\right).$$

∎

THEOREM 4. *If Assumption 1 holds and (A) is strictly stable, then (B) is also.*

PROOF. From Theorem 1 and our hypothesis on (A), $\|X_jX_i^{-1}\|$ is bounded on $\mathcal{N} \times \mathcal{N}$. We will show that $\|Y_jY_i^{-1}\|$ is also bounded on $\mathcal{N} \times \mathcal{N}$. Then Theorem 1 will imply the conclusion.

The argument used in the proof of Theorem 3 (with $\rho = 1$) shows that

$$\left\|Y_jY_i^{-1}\right\| \le \alpha\exp\left(\beta\sum_{m=0}^{\infty}\left\|F_m^{(p)}\right\|\right), \qquad i_0 \le i \le j.$$

Now suppose that $0 \le j \le i$. Summing (6) from $m = j$ to $m = i-1$ yields

$$X_j^{-1}S_jY_j = X_i^{-1}S_iY_i - \sum_{m=j}^{i-1} X_{m+1}^{-1}F_m^{(0)}Y_m,$$

and an argument like the one used to prove Lemma 1 yields

$$\Gamma_j^{(p)} Y_j = S_j^{-1} X_j \left[X_i^{-1} S_i \Gamma_i^{(p)} Y_i - \sum_{m=j}^{i-1} X_{m+1}^{-1} F_m^{(p)} Y_m \right], \qquad 0 \le j \le i.$$

Now an argument similar to the one that led to (10) shows that there are constants α_1 and β_1 such

$$\left\| Y_j Y_i^{-1} \right\| \le v_{ij}, \qquad i_0 \le j \le i \tag{13}$$

(recall that $(\Gamma_i^{(p)})^{-1}$ is bounded for $i \ge i_0$), with

$$v_{ij} = \alpha_1 + \beta_1 \sum_{m=j}^{i-1} \left\| F_m^{(p)} \right\| \left\| Y_m Y_i^{-1} \right\|, \qquad i_0 \le j \le i.$$

Now

$$v_{i,j+1} - v_{ij} = -\beta_1 \left\| F_j^{(p)} \right\| \left\| Y_j Y_i^{-1} \right\| \ge -\beta_1 \left\| F_j^{(p)} \right\| v_{ij}, \qquad i_0 \le j \le i-1,$$

so

$$v_{i,j+1} \ge \left(1 - \beta_1 \left\| F_j^{(p)} \right\| \right) v_{ij}, \qquad j_0 \le j \le i-1. \tag{14}$$

Now choose $j_0 \ge i_0$ so that $\beta_1 \|F_j^{(p)}\| < 1/2$, $j \ge j_0$. Then

$$\frac{1}{1 - \beta_1 \left\| F_j^{(p)} \right\|} \le 1 + \beta_1 \left\| F_j^{(p)} \right\| + 2\beta_1^2 \left\| F_j^{(p)} \right\|^2 \le \exp\left(\beta_1 \left\| F_j^{(p)} \right\| + 2\beta_1^2 \left\| F_j^{(p)} \right\|^2 \right),$$

so (14) implies that

$$v_{ij} \le v_{i,j+1} \exp\left(\beta_1 \left\| F_j^{(p)} \right\| + 2\beta_1^2 \left\| F_j^{(p)} \right\|^2 \right), \qquad j_0 \le j \le i-1.$$

Since $v_{ii} = \alpha_1$ this implies that

$$v_{ij} \le \alpha_1 \prod_{m=j}^{i-1} \exp\left(\beta_1 \left\| F_m^{(p)} \right\| + 2\beta_1^2 \left\| F_m^{(p)} \right\|^2 \right), \qquad j_0 \le j \le i.$$

This and (13) imply that

$$\left\| Y_j Y_i^{-1} \right\| \le \alpha_1 \exp\left(\beta_1 \sum_{m=0}^{\infty} \left(\left\| F_m^{(p)} \right\| + 2\beta_1 \left\| F_m^{(p)} \right\|^2 \right) \right), \qquad j_0 \le j \le i,$$

which completes the proof. ∎

THEOREM 5. *Suppose that Assumption 1 holds and $\lim_{j\to\infty} S_j = S_\infty$ exists. Then, if (A) has linear asymptotic equilibrium so does (B).*

PROOF. Since $\lim_{j\to\infty} X_j$ is invertible, X and X^{-1} are bounded. Therefore (A) is strictly stable, by Theorem 1. Now Theorem 4 implies that (B) is strictly stable; in particular, Y^{-1} is bounded, by Theorem 1. Our assumptions on S imply that S_∞ is invertible and that $\lim_{j\to\infty} S_j^{-1} = S_\infty^{-1}$. Since $\lim_{j\to\infty} \Gamma_j^{(p)} = I$ and $F^{(p)} \in \mathcal{A}$, (5) implies that $\lim_{j\to\infty} Y_j$ exists. This limit must be invertible, since Y^{-1} is bounded. Therefore (B) has linear asymptotic equilibrium, by Theorem 1. ∎

COROLLARY 1. *Suppose that there is an $S \in \mathcal{S}$ such that the function defined by*

$$F_m^{(0)} = \Delta S_m + S_{m+1} B_m \tag{15}$$

is in $\mathcal{A}$, or it is in $\mathcal{I}$ and there is a positive integer p such that the $n \times n$ matrix functions $F^{(1)}, \dots, F^{(p)}$ defined by

$$F_m^{(r)} = \left(\sum_{k=m+1}^{\infty} F_k^{(r-1)} \right) B_m, \qquad 1 \le r \le p, \tag{16}$$

are in $\mathcal{I}$, and $F^{(p)} \in \mathcal{A}$. Then (B) is strictly stable. Moreover, if $\lim_{j\to\infty} S_j$ exists then (B) has linear asymptotic equilibrium.

PROOF. The functions $F^{(0)}$ and $F^{(r)}$ defined by (15) and (16) are the same as those defined by (3) and (4) if $A = 0$. Since the system $\Delta x_m = 0$ clearly has linear asymptotic equilibrium and is therefore strictly stable, Theorems 4 and 5 imply the conclusions. ∎

Applying Corollary 1 with $S_m = I$ yields the following corollary.

COROLLARY 2. *Suppose that*

$$\sum_{m=0}^{\infty} \|B_m\| < \infty \tag{17}$$

or there is an integer $p \ge 1$ such that the sums

$$B_m^{(r)} = \left(\sum_{k=m+1}^{\infty} B_k^{(r-1)} \right) B_m, \qquad 1 \le r \le p,$$

(with $B_j^{(0)} = B_j$) converge, and

$$\sum_{m=0}^{\infty} \left\| B_m^{(p)} \right\| < \infty. \tag{18}$$

Then (B) has linear asymptotic equilibrium.

It is known (see, for example [4,5]) that (17) is a sufficient conditions for (B) to have linear asymptotic equilibrium. In [4] the author showed that (18) with $p = 1$ is a sufficient condition. In its full generality Corollary 2 is an analog of a result of Wintner [6] for a linear differential system $y' = B(t)y$.

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